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THE DEVELOPMENT OF FUNCTIONS.

BY S. A. COREY.

THE development of functions into series has been effected to a large extent, as is well known, by the use of Taylor's, Maclaurin's and Lagrange's formulas. There are, however, a large number of cases where these formulas are of no practical value in determining the function sought. In some cases the resulting series, while theoretically convergent, become practically useless on account of the complicated character of the successive derivatives of the function. In many other important cases the resulting series do not converge, or converge too slowly to be of service for numerical computation.

It is the purpose of this paper to call attention to other formulas, similar in some respects to the older formulas, but more rapidly convergent. As far as I am aware these formulas are new. If I am in error on this point I should be glad to be so informed.

Using the notation ordinarily employed in Taylor's and Maclaurin's formulas, these new formulas may be written:

$$\begin{aligned} f(a+x)=f(a)&+\frac{x}{2}[f'(a)+f'(a+x)]+\frac{x^2}{2^2.2!}[f''(a)-f''(a+x)] \\ &+\frac{x^3}{2^3.3!}[f'''(a)+f'''(a+x)]+\frac{x^4}{2^4.4!}[f^{(4)}(a)-f^{(4)}(a+x)] \\ &+\frac{x^5}{2^5.5!}[f^{(5)}(a)+f^{(5)}(a+x)]+\cdots\cdots. \end{aligned} \qquad [\mathrm{I}]$$

$$f(a+2x)=f(a)+2\left[\frac{x}{1}f'(a+x)+\frac{x^3}{3!}f'''(a+x)+\frac{x^5}{5!}f^{(5)}(a+x)+\cdots\cdot\right]. \qquad [\mathrm{II}]$$

$$\begin{aligned} f(a+x)=f(a)&+\frac{x}{2m}\Big\{f'(a)+f'(a+x)+2\Big[f'\Big(a+\frac{x}{m}\Big)+f'\Big(a+\frac{2x}{m}\Big)+\cdots\cdot \\ &\qquad\qquad +f'\Big(a+\frac{(m-1)x}{m}\Big)\Big]\Big\} \\ &+\frac{x^2}{2^2 2!m^2}\Big[f''(a)-f''(a+x)\Big]+\frac{x^3}{2^3 3!m^3}\Big\{f'''(a)+f'''(a+x) \\ &+2\Big[f'''\Big(a+\frac{x}{m}\Big)+f'''\Big(a+\frac{2x}{m}\Big)+\cdots+f'''\Big(a+\frac{(m-1)x}{m}\Big)\Big]\Big\} \\ &+\frac{x^4}{2^4 4!m^4}\Big[f^{(4)}(a)-f^{(4)}(a+x)\Big]+\cdots\cdots \end{aligned} \qquad [\mathrm{III}]$$

$$
\begin{aligned}
f(a+2mx)=f(a)&+\frac{2x}{1}\left\{f'(a+x)+f'(a+3x)+\cdots\cdots+f'[a+(2m-1)x]\right\}\\
&+\frac{2x^3}{3\,!}\left\{f'''(a+x)+f'''(a+3x)+\cdots\cdots+f'''[a+(2m-1)x]\right\}\\
&+\frac{2x^5}{5\,!}\left\{f^{(5)}(a+x)+f^{(5)}(a+3x)+\cdots\right.\\
&\qquad\qquad\left.+f^{(5)}[a+(2m-1)x]\right\}+\cdots \qquad [\mathrm{IV}]
\end{aligned}
$$

Formula (I) can be verified by developing $f'(a+x), f''(a+x)$, etc., in the second member into power series by Taylor's Theorem and combining terms including like powers of x. It at once reduces to the familiar development of $f(a+x)$.

If both members of formula (II) are differentiated with respect to x there will result the Taylor's development of $f'[(a+x)+x]$.

Formula (III) may be inferred from (I) without much labor by a step to step process, *i. e.* by developing by (I):

$$f\left(a+\frac{x}{m}\right),\ f\left[\left(a+\frac{x}{m}\right)+\frac{x}{m}\right],\ f\left[\left(a+\frac{2x}{m}\right)+\frac{x}{m}\right]\text{etc.,}$$

simplifying each by the aid of the preceding.

Formula (IV) may be deduced by a similar step by step process from (II).

It is to be observed that the developments given by (I), (II), (III), and (IV) are not power series, and while they are perfectly trustworthy for purposes of computation whenever the Taylor's development, on which they ultimately depend, is valid, it is by no means clear that they may be safely used for other purposes; for instance that their term by term integrals or derivatives will be equal to the integral or derivative of the function.

That formulas (I) and (II) converge more rapidly than Taylor's Formula is easily seen by comparing the general term in each with the corresponding term of Taylor's Formula. The general term of (I) is

$$\frac{x^n}{2^n n\,!}\left[f^{(n)}(a)-(-1)^n f^{(n)}(a+x)\right],$$

of Taylor's Formula is $\frac{x^n}{n\,!}f^{(n)}(a)$.

The general term of (II) is $\left[1-(-1)^n\right]\dfrac{x}{n!}f^{(n)}(a+x)$, the corresponding term of Taylor's Formula is $\dfrac{2^n x^n}{n!}f^{(n)}(a)$.

Formulas (III) and (IV) can be made to converge as rapidly as may be desired by taking a sufficiently large value of m, and are especially valuable when the labor of calculating the successive derivatives of the function increases rapidly with the increase in the order of the derivative.

The following are some of the developments obtained by formulas (I) and (II):

$$(a+x)^n = a^n + \frac{n[a^{n-1}+(a+x)^{n-1}]}{2}x + \frac{n(n-1)[a^{n-2}-(a+x)^{n-2}]}{8}x^2 + \frac{n(n-1)(n-2)[a^{n-3}+(a+x)^{n-3}]}{2^3\,3!}x^3 + \cdots\cdots \tag{1}$$

If n is a positive whole number and odd the second member of (1) will terminate with the nth power of x. If n is an even whole number and positive the second member will terminate with the $(n-1)$th power of x.

If n is equal to $1/m$ we get, after reduction, the following expression for the m th root of $(a+x)$:

$$(a+x)^{\frac{1}{m}} = a^{\frac{1}{m}}\left[\frac{1+\dfrac{x}{2ma}-\dfrac{(m-1)x^2}{8m^2a^2}+\dfrac{(1-3m+2m^2)}{48m^3a^3}x^3-\cdots\cdots}{1-\dfrac{x}{2ma}-\dfrac{(m-1)x^2}{8m^2a^2}-\dfrac{(1-3m+2m^2)}{48m^3a^3}x^3-\cdots\cdots}\right]. \tag{2}$$

If, as a special case, we let $a=32$, $x=-2$, and $m=5$, we get (by substituting in the foregoing expression) 1.97435047 as the fifth root of 30.

Expanding $\log(1+x)$ by formula (I) we have:

$$\log(1+x) = \frac{1+\dfrac{1}{1+x}}{2}x - \frac{1-\dfrac{1}{(1+x)^2}}{8}x^2 + \frac{2+\dfrac{2}{(1+x)^3}}{48}x^3 - \cdots\cdots \tag{3}$$

Expanding by formula (II):

$$\log(1+x) = 2\left[\frac{x}{2+x}+\frac{x^3}{3(2+x)^3}+\frac{x^5}{5(2+x)^5}+\cdots\cdot\right]. \tag{4}$$

It may be noted that (4) is valid for any value of x.

Expanding e^x by formula (I) and reducing we have:

$$e^x = \frac{1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{2^4 4\,!} + \cdots\cdots}{1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} - \frac{x^4}{2^4 4\,!} - \cdots\cdots}. \tag{5}$$

Expanding e^{2x} by formula (II) :

$$e^{2x} = 1 + 2\,e^x \left(x + \frac{x^3}{3\,!} + \frac{x^5}{5\,!} + \frac{x^7}{7\,!} + \cdots\cdots \right). \tag{6}$$

Expanding sin x by formula (I) and reducing:

$$\sin x = \frac{\cos x \left(\frac{x}{2} - \frac{x^3}{48} + \frac{x^5}{2^5 5\,!} - \cdots\cdots \right) + \left(\frac{x}{2} - \frac{x^3}{48} + \frac{x^5}{2^5 5\,!} - \cdots\cdots \right)}{1 - \frac{x^2}{8} + \frac{x^4}{2^4 4\,!} - \frac{x^6}{2^6 6\,!} + \cdots\cdots}. \tag{7}$$

Elliptic and hyperelliptic integrals and a large class of Abelian integrals may be represented by:

$$\int \frac{dx}{(X)^r}$$

where X is some polynomial in x, and r some fraction. In this class of functions the successive derivatives are usually very complicated, and formula (III) or formula (IV) will frequently be useful.

I will give but one example, namely, the evaluation of the well-known elliptic integral,

$$\int_0^x \frac{dx}{[(1 - k^2x^2)(1 - x^2)]^{\frac{1}{2}}}\,, \text{ where } x = \sqrt{\tfrac{1}{2}}, \text{ and } k = \sqrt{\tfrac{1}{2}}.$$

Using but two terms of formula (III), and making m equal to 10, we get 825+ as the value of the integral; using four terms with the same value of m, we get .82605, accurate to the last place of decimals.

Formula (IV) might have been employed in this case with equal accuracy.

In a process of approximation, such as that described by W. E. Durand, in THE ANNALS OF MATHEMATICS, 1st Series, Vol. 12, p. 110, formulas (I), (II), (III), and (IV) can usually be employed with decided advantage.

Cases might be enumerated indefinitely to show the developments obtainable by the formulas herein given, and to show the rapid convergence of the resulting series in the majority of cases, but such enumeration of cases, while interesting and instructive, would unduly lengthen this paper. Enough has been said, I believe, to arouse the interest of working mathematicians in the formulas given and to induce them to further study and develop the methods herein employed.

HITEMAN, IOWA, OCTOBER, 1899.